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***On the Reduction of Hyperelliptic Integrals ($p=3$) to
Elliptic Integrals by Transformations of
the Second and Third Degrees.***

BY WILLIAM GILESPIE.

INTRODUCTION.

The principal subject of this paper is an application of cubic involution to the problem of the reduction to elliptic integrals, of hyperelliptic integrals of genus $p=3$ and of the first kind, by a rational transformation of the third degree.

It forms thus a continuation of Professor Bolza's researches on the cubic transformations of elliptic integrals and cubic reductions of hyperelliptics of genus $p=2$.*

It is also closely connected with the work of J. I. Hutchinson in his dissertation† where he applies the quadratic involution to a similar problem where $p=2$.

In Part I, the general case of cubic reduction is considered after the connection of the problem with the theory of cubic involution has been established (§1).

The invariant condition which must be satisfied, in the case of cubic reduction, by the octavic on whose square root the hyperelliptic integral depends, is obtained (§2), and the most general reducible integral is obtained in four general forms, the last two being independent of the cubic involution (§§6 and 7).

The main subject of Part II is the singular case of simultaneous reduction of the second and third degree; that is, where two hyperelliptic integrals having the same octavic are reducible, one by a quadratic transformation and the other

* "Die cubische Involution und Dreitheilung," etc., and "Zur Reduction Hyperelliptischer Integrals," etc., Math. An., Bd. 50, pp. 68 and 314.

† "On the Reduction of Hyperelliptic Functions ($p=2$)," etc., Chicago, 1897.

by a transformation of the third degree (§3-§8). As an introduction to this part, the general case of quadratic reduction ($p=3$) is considered (§1), and the special cases of more than one reducible integral with a given radical, are determined. A complete table of such cases is formed (§2).

In the last part, the involution is specialized to one containing two cubes. For this special form, the most general reducible integral is determined (§1) and the cases of simultaneous reduction of the second and third degree (§2). For one of the latter cases, in which the hyperelliptic curve can be reduced to the normal form

$$y^2 = x(1 - x^6),$$

the reduction problem is studied by means of the Weierstrass-Picard theorem on the periods of reducible integrals (§3). This leads not only to a confirmation of the results previously obtained by algebraic methods, but also to a determination of all hyperelliptic integrals of the first kind belonging to the curve

$$y^2 = x(1 - x^6),$$

which are reducible to elliptic integrals by rational transformations of any degree.

In the following review but the merest outline of the work can be given; further details of the work may be found in the dissertation on this subject (Chicago, 1899).

PART I.

§1.—*The Connection of the Problem with Cubic Involution.*

The method used by Jacobi* for the transformation of elliptic integrals, can be extended at once to the investigation of the reducibility of hyperelliptics, and furnishes the results:

In order that the hyperelliptic integral of the first kind and of genus $p=3$,

$$\int \frac{q(x_1 x_2)(xdx)}{\sqrt{R(x_1 x_2)}} \quad (1)$$

(where $q(x_1 x_2)$ is a quadratic and $R(x_1 x_2)$ an octavic quantic), *shall be reducible to the elliptic integral*

$$m \int \frac{(ydy)}{\sqrt{(y_1 - \lambda_1 y_2)(y_1 - \lambda_2 y_2)(y_1 - \lambda_3 y_2)(y_1 - \lambda_4 y_2)}}. \quad (2)$$

* "Fundamenta Nova," Werke, Bd. I, p. 55.

m being a constant by the rational transformation

$$\left. \begin{aligned} y_1 &= U, \\ y_2 &= V, \end{aligned} \right\} \quad (3)$$

where U and V are cubic quantics in $(x_1 x_2)$, it is necessary and sufficient, first, that R can be broken up in two cubic factors R_1 and R_2 and a quadratic $(xe_1)(xe_2)$ such that

$$\left. \begin{aligned} U - \lambda_1 V &= R_1(x_1 x_2), \\ U - \lambda_2 V &= R_2(x_1 x_2), \\ U - \lambda_3 V &= (xd_1)^2(xe_1), \\ U - \lambda_4 V &= (xd_2)^2(xe_2), \end{aligned} \right\} \quad (4)$$

or, what is the same, R must be decomposable into three factors, two cubics which shall define a cubic involution and a quadratic whose roots are branch points of that involution; and secondly, since $(ydy) = UdV - VdU$, which is proportional to the Jacobian of U and V whose roots are the double points of the involution, the roots of the quadratic q must be the two double points not corresponding to the branch points in R .

Thus a reducible integral must have the form

$$\int \frac{(xd_3)(xd_4)(xdx)}{\sqrt{(xe_1)(xe_2)(\lambda_1 f_1 + \lambda_2 f_2)(\mu_1 f_1 + \mu_2 f_2)}}, \quad (5)$$

where f_1 and f_2 are linear homogeneous functions of the cubics U and V and $(xd_i)^2(xe_i)$ for $i = 1, 2, 3, 4$, denote the branch triples of the involution $U + \lambda V$.

If an integral have this form, it is reducible.

§2.—The Cubic Involution and the Invariant Condition for Reducibility.

We collect in this paragraph those properties of the cubic involution of which we will make frequent use.

A cubic involution is defined by the equation

$$f_1 + \lambda f_2 = 0, \quad (6)$$

where λ is an arbitrary parameter and f_1 and f_2 two cubic quantics.

We will require the following combinants

$$\mathfrak{S} = (f_1 f_2)_1, \quad J = (f_1 f_2)_3, \quad H = (\mathfrak{S} \mathfrak{S})_2, \quad i_s = (\mathfrak{S} \mathfrak{S})_4, \quad j = (\mathfrak{S} H)_4, \quad (7)$$

Being given one point (y) of a triple, the other points are the roots of the affinity equation

$$3\mathfrak{S}_x^2 \mathfrak{S}_y^2 + \frac{1}{2} J(xy)^2 = 0. \quad (8)$$

The double points of the involution are the roots of $\mathfrak{S}$, while the branch points are those of

$$3H_{\mathfrak{S}} + J\mathfrak{S}. \quad (9)$$

The first polar of

$$\Gamma = 2H_{\mathfrak{S}} + \frac{1}{3}J\mathfrak{S}, \quad (10)$$

with respect to an arbitrary parameter λ , is the cubic involution itself. For a given $\mathfrak{S}$ there exists, besides the involution spoken of above, another called the conjugate involution. For this involution all the statements made above hold good if we first change the sign of J .*

The condition which must be satisfied in the case of cubic reduction (§1) by the octavic R can be expressed thus:

In order that there shall exist a reducible hyperelliptic of the first kind belonging to R , R must be decomposable into the factors R_1 , R_2 and N , where N is a quadratic dividing exactly the branch form $3H_{\mathfrak{S}} + J\mathfrak{S}$ for the involution determined by the cubic factors R_1 and R_2 .

This is equivalent to two algebraic relations between the roots of R . In accordance with the general theory of reduction,† it can be represented by the identical vanishing of a simultaneous biquadratic invariant of N and $3H + J\mathfrak{S}$.‡

§3.—*A Normal Form for the Reducible Integral.*

From the form of the reducible integral in §1, we see that the double points are associated in pairs, and for that reason we choose the following normal form used by Klein in his "Modulfunctionen" (p. 7):

Denoting the roots of $\mathfrak{S}$ by

$$x_1^{(i)} : x_2^{(i)} \quad i = 1, 2, 3, 4 \quad \text{and} \quad x_1^{(i)} x_2^{(k)} - x_2^{(i)} x_1^{(k)} \quad \text{by} \quad (ik),$$

we have the invariants

$$A = (12)(34); \quad B = (13)(42); \quad C = (14)(23), \quad (11)$$

where

$$A + B + C = 0.$$

* We refer for these statements to Bolza's paper in the *Annalen*, Bd. 50, §1, where the literature of the subject is given.

† Koenigsberger, *Crelle*, 85.

‡ Clebsch, "Binäre Formen," §27, p. 94.

Then, by a linear transformation of determinant 1, $\mathfrak{S}$ can be transformed into the form

$$(\mathfrak{S}) \quad x_1 x_2 (x_2 - x_1)(Ax_1 + Bx_2). \quad (12)$$

Here the pairing of the roots is put in evidence. The invariants may be expressed in the following manner:

$$A = 2i\sqrt{3} l_1 \bar{l}_1; \quad B = 2i\sqrt{3} l_2 \bar{l}_2; \quad C = 2i\sqrt{3} l_3 \bar{l}_3, \quad (13)$$

where the l 's are the irrational invariants of the cubic involution used by Bolza* and are connected by the following relations:

$$l_1 + l_2 + l_3 = 0; \quad l_1 - l_2 = i\sqrt{3} \bar{l}_3; \quad l_2 - l_3 = i\sqrt{3} \bar{l}_1; \quad l_3 - l_1 = i\sqrt{3} \bar{l}_2. \quad (14)$$

Using the affinity equation and these values for the invariants, we find the following branch triples:

$$\begin{array}{ll} \text{double point.} & \text{branch point.} \\ \text{I. } x_1 = 0 & x_1 = i\sqrt{3} \frac{\bar{l}_2}{l_1} x_2; \\ \text{II. } x_2 = 0 & x_1 = \frac{l_2}{i\sqrt{3} \bar{l}_1} x_2; \\ \text{III. } x_1 = x_2 & x_1 = \frac{l_2}{l_1} x_2; \\ \text{IV. } x_1 = -\frac{l_2 \bar{l}_2}{l_1 \bar{l}_1} x_2 & x_1 = -\frac{\bar{l}_2}{l_1} x_2. \end{array} \quad (15)$$

Then, using the first two double points as roots of q and the first two branch triples as U and V , we obtain the result:

Every hyperelliptic integral which is reducible by a transformation of the third degree, can be transformed by a linear transformation of the independent variable into the form

$$2i\sqrt{3} \int \frac{x_1 x_2 (x dx)}{\sqrt{(l_1 x_1 - l_2 x_2)(\bar{l}_1 x_1 + \bar{l}_2 x_2)[\lambda_1 (l_1 x_1 + i\sqrt{3} \bar{l}_2 x_2) x_1^2 + \lambda_2 (i\sqrt{3} l_1 x_1 - l_2 x_2) x_2^2]}}, \quad (16)$$

$$\times [\mu_1 (l_1 x_1 + i\sqrt{3} \bar{l}_2 x_2) x_1^2 + \mu_2 (i\sqrt{3} l_1 x_1 - l_2 x_2) x_2^2]$$

which, by the transformation

$$y_1 = (l_1 x_1 + i\sqrt{3} \bar{l}_2 x_2) x_1^2, \quad y_2 = (i\sqrt{3} l_1 x_1 - l_2 x_2) x_2^2,$$

* L. c., equation (77).

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reduces to the elliptic integral

$$\int \frac{(y dy)}{\sqrt{(y_1 + y_2)(l_1 \bar{l}_1^3 y_1 - l_2 \bar{l}_2^3 y_2)(\lambda_1 y_1 + \lambda_2 y_2)(\mu_1 y_1 + \mu_2 y_2)}}, \quad (17)$$

where $\lambda_1 : \lambda_2$ and $\mu_1 : \mu_2$ are arbitrary parameters and the l 's are defined above; the integral thus depends on three independent non-homogeneous parameters, which agrees with the result in the end of §2.

§4.—A Theorem on Cubic Involutions.

The following lemma will help us to obtain the reducible integral in its most general form, independent of any normal form.

We have from Clebsch* immediately

$$\mathfrak{S} = \frac{2}{m_2 - m_3} (\chi^2 - \psi^2), \quad (18)$$

where ψ and χ are two of the quadratic factors of T_3 and the m 's are roots of the cubic resolvent of $\mathfrak{S}$.

We deduce:

$$\text{(branch equation)} \quad 3H_3 + J\mathfrak{S} = \frac{6}{m_2 - m_3} (l_3^2 \psi^2 - l_2^2 \chi^2), \quad (19)$$

$$\Gamma = 2H + \frac{1}{3}J\mathfrak{S} = \frac{4\bar{l}_1}{m_2 - m_3} (\bar{l}_2 \psi^2 - \bar{l}_3 \chi^2), \quad (20)$$

$$\text{(affinity equation)} \quad \frac{2}{m_2 - m_3} [\chi(x)\chi(y) - \psi(x)\psi(y)] + l_2 l_3 (xy)^2, \quad (21)$$

where $(xy) = x_1 y_2 - x_2 y_1$.

If we form the product of the affinity equations for the two roots of the factor $l_3 \psi - l_2 \chi$ of the branch equation, we find it has the form

$$l_1^2 l_2^2 l_3^2 (\psi - \chi)^2.$$

Thus we obtain the following lemma:

In any cubic involution the Jacobian and the branch equation, when expressed in terms of the Clebsch functions ψ and χ and the invariants m 's and l 's, assume the forms $\frac{2}{m_2 - m_3} (\chi^2 - \psi^2)$ and $\frac{6}{m_2 - m_3} (l_3^2 \psi^2 - l_2^2 \chi^2)$ respectively; the double points yielded by the factor $(\psi - \chi)$ of the Jacobian correspond to the branch points

* "Binäre Formen," p. 159.

yielded by the factor $(l_3 \psi - l_2 \chi)$ of the branch equation, and there is a similar correspondence between the roots of $(\psi + \chi)$ and $(l_3 \psi + l_2 \chi)$.

For the conjugate involution, the factors $(\psi - \chi)$ and $(\bar{l}_3 \psi - \bar{l}_2 \chi)$ correspond, and also $(\psi + \chi)$ and $(\bar{l}_3 \psi + \bar{l}_2 \chi)$.

§5.—First General Form of the Reducible Integral.

This lemma of (§4) will be applied now to the form of the reducible integrals given in equation (5).

We may suppose

$$\left. \begin{aligned} (\psi + \chi) &= (xd_1)(xd_2); & (\psi - \chi) &= (xd_3)(xd_4), \\ (l_3 \psi + l_2 \chi) &= (xe_1)(xe_2); & (l_3 \psi - l_2 \chi) &= (xe_3)(xe_4), \\ (\bar{l}_3 \psi + \bar{l}_2 \chi) &= (x\bar{e}_1)(x\bar{e}_2); & (\bar{l}_3 \psi - \bar{l}_2 \chi) &= (x\bar{e}_3)(x\bar{e}_4). \end{aligned} \right\} \quad (22)$$

This, however, involves an agreement in the choice of the homogeneous coordinates of the points $d_i e_i$ and $\bar{e}_i$.

Also, since Γ or $\frac{4i}{\sqrt{3}\bar{l}_1} (\bar{l}_2 \psi^2 - \bar{l}_3 \chi^2)$ has for its first polar with regard to an arbitrary parameter the cubic involution itself $(\Gamma_x^3 \Gamma_\lambda)$, $\therefore$ the reducible integral (5) has the form

$$\int \frac{(\psi - \chi)(xdx)}{\sqrt{(l_3 \psi + l_2 \chi) \Gamma_x^3 \Gamma_\lambda \Gamma_x^{\prime 3} \Gamma_\mu^{\prime}}} \quad (23)$$

The transformation

$$y_1 = U, \quad y_2 = V$$

can, by a linear transformation of the elliptic integral, be thrown into the form $\Gamma_x^3 \Gamma_y = 0$ or

$$y_1 = \frac{1}{4} \frac{\partial \Gamma}{\partial x_2}, \quad y_2 = -\frac{1}{4} \frac{\partial \Gamma}{\partial x_1}. \quad (24)$$

Let us now consider the elliptic integral.

Before reduction, the hyperelliptic integral has the form

$$\int \frac{\mathfrak{S}(xdx)}{\sqrt{(xd_1)^2 (xe_1)(xd_2)^2 (xe_2) \Gamma_x^3 \Gamma_\lambda \Gamma_x^{\prime 3} \Gamma_\mu^{\prime}}}, \quad (25)$$

and by the transformation (24)

$$\left. \begin{aligned} (y\bar{e}_i) &= \Gamma_x^3 \Gamma_{e_i} = \rho_i (xe_i)(xd_i)^2, & i &= 1, 2, 3, 4, & [\text{Bolza, l. c. (34)}] \\ (y\lambda) &= \Gamma_x^3 \Gamma_\lambda, & \text{let } (y\lambda)(y\mu) &= g(y), \end{aligned} \right\} \quad (26)$$

we also have

$$(ydy) = \frac{3}{2} \Omega \mathfrak{S}(xdx), \quad [\text{Bolza, l. c. (38)}]$$

$\therefore$ after the transformation, our integral has the form

$$C \int \frac{(y dy)}{\sqrt{(\bar{l}_3 \psi(y) + \bar{l}_2 \chi(y)) g(y)}}, \quad (27)$$

where C is a constant and g is an arbitrary quadratic having as roots the parameters λ and μ .

Returning to the hyperelliptic integral (23), since

$$\Gamma_x^3 \Gamma_x'^3 \Gamma_\lambda \Gamma_\mu' = (\Gamma g)_2 \Gamma - \frac{1}{2} H_\Gamma g, \quad (28)$$

it must have the form

$$\int \frac{(\psi - \chi)(x dx)}{\sqrt{(l_3 \psi + l_2 \chi)[(\Gamma g)_2 \Gamma - \frac{1}{2} H_\Gamma g]}}. \quad (29)$$

Equations (24) and (27) give us the transformation and the elliptic integral, but the constant factor must be obtained.

To find this factor, we must collect those from the following equations:

$$(y dy) = \frac{2}{3} \Omega \cdot \mathfrak{D} \cdot (x dx), \quad \text{see (26)}, \quad (30)$$

$$\Gamma_x^3 \Gamma_{e_1} \Gamma_x'^3 \Gamma_{e_2}' = \frac{\Omega}{l_1 \bar{l}_2} (l_3 \psi + l_2 \chi)(\psi + \chi)^2, \quad (31)$$

where $\sqrt{\Omega} = 2\bar{l}_1 \bar{l}_2 \bar{l}_3$ and its conjugate $\sqrt{\Omega} = 2l_1 l_2 l_3$, [Bolza, l. c., §7], (32)

and also the factor from

$$\mathfrak{D} = \frac{2}{i\sqrt{3}l_1 \bar{l}_1} (\psi^2 - \chi^2), \quad \text{see (18)}, \quad (33)$$

we have thus the theorem:

The hyperelliptic integral

$$\frac{\sqrt{3\Omega}}{i\sqrt{\bar{l}_1 \bar{l}_1}} \int \frac{(\psi - \chi)(x dx)}{\sqrt{(l_3 \psi + l_2 \chi)[(g\Gamma)_2 \Gamma - \frac{1}{2} H_\Gamma g]}}, \quad (34)$$

by the transformation $\rho y_1 = \frac{1}{4} \frac{\partial \Gamma}{\partial x_2}$, $\rho y_2 = -\frac{1}{4} \frac{\partial \Gamma}{\partial x_1}$ reduces to the elliptic integral

$$\int \frac{(y dy)}{\sqrt{(\bar{l}_3 \psi(y) + \bar{l}_2 \chi(y)) g(y)}}. \quad (35)$$

§6.—*A Second General Form of the Reducible Integral.*

It is possible to express these integrals and the transformation in terms of the biquadratic Γ , so that, knowing Γ , we can find the integral.

We know that $T_\phi = 2\phi\psi\chi$, [Clebsch, l. c., p. 159].

Let now $T_\Gamma = 2\Phi\Psi X$, (36)

and let n_1, n_2, n_3 be the roots of the resolvent of Γ . Then it is found that

$$\phi = \frac{1}{\sqrt{2n_1}} \Phi, \quad \psi = \frac{1}{\sqrt{2n_2}} \Psi, \quad \chi = \frac{1}{\sqrt{2n_3}} X, \quad (37)$$

and using the relations* that exist between the l 's, n 's and Ω , we obtain the reducible integral (34) in a new form, whence the theorem:

If Γ be any biquadratic quantic and Φ, Ψ and X the three quadratic factors of T_Γ , n_1, n_2 and n_3 , the roots of the resolvent of Γ and g an arbitrary quadratic, then the most general reducible hyperelliptic integral has the form

$$\frac{3}{(n_2 - n_3)^{\frac{1}{2}}} \int \frac{(\sqrt{n_3} \Psi - \sqrt{n_2} X)(xdx)}{\sqrt{[\sqrt{n_3}(n_1 - n_2)) \Psi + \sqrt{n_2}(n_3 - n_1) X][(\Gamma g)_2 \Gamma - \frac{1}{2} H_\Gamma g]}} \quad (38)$$

and reduces to the elliptic integral

$$\int \frac{(ydy)}{\sqrt{[\sqrt{n_3} \Psi(y) + \sqrt{n_2} X(y)] g(y)}} \text{ by the transformation } \begin{cases} y_1 = \frac{1}{4} \frac{\partial \Gamma}{\partial x_2}, \\ y_2 = -\frac{1}{4} \frac{\partial \Gamma}{\partial x_1} \end{cases}$$

This form is completely independent of the cubic involution.

§7.—*A Third General Form of the Reducible Integral.*

Finally, the integral can be expressed in terms of three quadratics ξ, η and ζ , defined in the following way:

$$\left. \begin{aligned} \Phi &= \frac{i\sqrt{3}}{2} \Omega^{\frac{1}{2}} \sqrt{l_2 l_3} \xi, \\ \Psi &= \frac{i\sqrt{3}}{2} \Omega^{\frac{1}{2}} \sqrt{l_3 l_1} \eta, \\ X &= \frac{i\sqrt{3}}{2} \Omega^{\frac{1}{2}} \sqrt{l_1 l_2} \zeta, \end{aligned} \right\} \quad (39)$$

* Bolza, l. c., (85), (86).

Since the functions Φ , Ψ and X are mutually harmonic, it follows that ξ , η and ζ are also, and the discriminant of each can be shown to be equal to 2.

Substituting in these values in (38), we obtain the theorem:

If ξ , η , ζ be three quadratic quantics mutually harmonic and so normalized that the discriminant of each is 2, and if also $l_1, l_2, l_3, \bar{l}_1, \bar{l}_2, \bar{l}_3$ be any quantities which satisfy the four relations

$$l_1 + l_2 + l_3 = 0, \quad l_1 - l_2 = i\sqrt{3}\bar{l}_3, \quad l_2 - l_3 = i\sqrt{3}\bar{l}_1, \quad l_3 - l_1 = i\sqrt{3}\bar{l}_2,$$

and if g be an arbitrary quadratic, then the most general reducible hyperelliptic integral has the form

$$i\sqrt{3} \int \frac{[\sqrt{l_3\bar{l}_3}\eta - \sqrt{\bar{l}_3l_2}\zeta](xdx)}{\sqrt{(\sqrt{l_3^3\bar{l}_3}\eta + \sqrt{\bar{l}_2l_2}\zeta)[(l_3B\eta - l_2C\zeta)(l_3\eta^2 - l_2\zeta^2) - g(l_3^2\eta^2 + l_2^2\zeta^2)]}}, \quad (40)$$

(where B and C are constants equal to $(\eta g)_2$ and $(\zeta g)_2$ respectively), and reduces to the elliptic integral

$$\int \frac{(ydy)}{\sqrt{(\sqrt{l_3^3\bar{l}_3}\eta(y) + \sqrt{\bar{l}_2l_2}\zeta(y))g(y)}}$$

by the cubic transformation

$$l_3\eta_x'^2\eta_x\eta_y - l_2\zeta_x'^2\zeta_x\zeta_y = 0,$$

where $\eta_x'^2 = \eta_x'^2 = \eta(x)$ and $\zeta(x) = \zeta_x'^2 = \zeta'_x$.

This form is completely independent of the involution.

PART II.

§1.—General Case of Reduction of the Second Degree.

As an introduction to this part, we consider the general form of a hyperelliptic integral of the first kind and of genus 3, which is reducible by a rational transformation of the second degree.

Using again the method of Jacobi just as in Part I, §1, we deduce the theorem:

In order that the hyperelliptic integral

$$\int \frac{q(x_1x_2)(xdx)}{\sqrt{R(x_1x_2)}} \quad (41)$$

(where q is a quadratic and R an octavic quantic), shall be reducible to the elliptic integral

$$m \int \frac{y dy}{\sqrt{(y_1 - \lambda_1 y_2)(y_1 - \lambda_2 y_2)(y_1 - \lambda_3 y_2)(y_1 - \lambda_4 y_2)}},$$

m being a constant, by a rational transformation

$$\begin{aligned} y_1 &= U \\ y_2 &= V \end{aligned}$$

where U and V are quadratic quantics in $(x_1 x_2)$, it is necessary and sufficient that R shall be decomposable into four factors belonging to the same quadratic involution, and that the quantic q shall have as roots the double points of that involution.

By a parametric representation of the roots of R as points on a conic section, we have a geometrical statement for these conditions. The points of R must be so situated that we can pair them in such a way that the four joins shall pass through a common point, and the tangents to the conic from this point shall touch in the points which represent the roots of q .

§2.—Singular Cases of Reduction of the Second Degree.

From the conditions just determined, we conclude that in general for a given octavic R , which fulfills the required condition of §1, there exists but one reducible integral. Some octavics, however, are decomposable in the required way in more ways than one. In such cases, more than one reducible integral will exist with the same octavic R .

The singular cases can be determined by the method used by J. I. Hutchinson in his dissertation (Chicago, 1897). First determine all octavics invariant under the finite linear substitution groups, then, for each octavic, pick out all the transformations for period 2 which interchange the eight roots in pairs.

Having all these transformations, we can form all the reducible integrals. The stationary or double points of each transformation are the roots of q , the numerator of the integral, and the R is in each case the invariant octavic. We may use for the reducing transformation

$$y_1 = (x d_1)^2, \quad y_2 = (x d_2)^2,$$

where d_1 and d_2 are the stationary points of the transformation.

The following table contains an enumeration of all such octavics, the groups under which each is invariant, the stationary points and the number of distinct reducible hyperelliptic integrals which can be obtained for each octavic:

R.	Group.	Stationary points.	Number of reducible integrals.
I (a) $x_1^8 + 4ax_1^6x_2^2 + 6bx_1^4x_2^4 + 4cx_1^2x_2^6 + dx_2^8$	cyclic $n = 2$	$x_1x_2 = 0$	1
I (b) $x_1x_2(x_1^6 + ax_1^4x_2^2 + bx_1^2x_2^4 + cx_2^6)$	cyclic $n = 2$		none
II $x_1(x_1^7 + x_2^7)$	cyclic $n = 7$		none
III (a) $x_1^8 + ax_1^6x_2^2 + bx_1^4x_2^4 + ax_1^2x_2^6 + x_2^8$	dihedron $n = 2$	$x_1x_2 = 0; x_1^2 = x_2^2; x_1^2 = -x_2^2$	3
III (b) $x_1x_2(x_1^2 + x_2^2)(x_1^4 + ax_1^2x_2^2 + x_2^4)$	dihedron $n = 2$	$x_1^2 = x_2^2$	1
IV $x_1x_2(x_1^6 + ax_1^3x_2^3 + x_2^6)$	dihedron $n = 3$	$x_1^2 = \varepsilon^{\frac{2n}{3}\pi i} x_2^2; (n = 0. 1. 2)$	3
V $x_1^8 + ax_1^4x_2^4 + x_2^8$	dihedron $n = 4$	$x_1x_2 = 0; x_1^2 = \varepsilon^{\frac{n\pi i}{2}} x_2^2; (n = 0. 1. 2. 3)$	5
VI $x_1x_2(x_1^6 + x_2^6)$	dihedron $n = 6$	$x_1^2 = \varepsilon^{\frac{2n+1}{3}\pi i} x_2^2; (n = 0. 1. 2)$	3
VII $x_1^8 - x_2^8$	dihedron $n = 8$	$x_1x_2 = 0; x_1^2 = \varepsilon^{\frac{2n+1}{4}\pi i} x_2^2; (n = 0. 1. 2. 3)$	5
VIII $x_1^8 + 14x_1^4x_2^4 + x_2^8$	octahedron	$x_1x_2 = 0; x_1^2 = x_2^2; x_1^2 = -x_2^2$ $x_1^2 = ix_2^2; x_1^2 = -ix_2^2$ $x_1^2 - 2ix_1x_2 + x_2^2 = 0; x_1^2 - 2ix_1x_2 - x_2^2 = 0$ $x_1^2 + 2ix_1x_2 + x_2^2 = 0, x_1^2 + 2ix_1x_2 - x_2^2 = 0$	9

§3.—*Cases of Simultaneous Reduction of the Second and Third Degree.*

We will now turn our attention to the problem of the determination of all the cases where there exist two hyperelliptic integrals of the first kind and genus 3, which have the same $R(x_1 x_2)$ and where one is reducible by a transformation of the second degree, while the other is reducible by one of the third.

We consider this from a geometrical standpoint.

By means of a parametric representation of a point on a conic section (C_2), Weyr gave the following geometric interpretation of a cubic involution :

The points of any triple of the involution define a triangle inscribed in the conic (C_2) and circumscribed about a conic (I_2) called the Involution Conic.

The points of intersection of (C_2) and (I_2) are the branch points of the involution, and the double points are the points where the common tangents to (C_2) and (I_1) touch (C_2).

If we choose for the triangle of reference the self-conjugate triangle of (C_2) and (I_2), the conics assume the equations* |

$$\left. \begin{aligned} (C_2) \quad x^2 + y^2 + z^2 &= 0, \\ (I_2) \quad \frac{x^2}{l_1^2} + \frac{y^2}{l_2^2} + \frac{z^2}{l_3^2} &= 0, \end{aligned} \right\} \quad (43)$$

where the l 's are the invariants defined in Part I, (13).

The coordinates of the double points are

$$\rho x = \sqrt{l_1 \bar{l}_1}, \quad \rho y = \pm \sqrt{l_2 \bar{l}_2}, \quad \rho z = \pm \sqrt{l_3 \bar{l}_3}, \quad (44)$$

and those of the branch points are

$$\rho x = l_1 \sqrt{\bar{l}_1 l_1}, \quad \rho y = \pm l_2 \sqrt{\bar{l}_2 l_2}, \quad \rho z = \pm l_3 \sqrt{\bar{l}_3 l_3}. \quad (45)$$

If, now, $R(x_1 x_2)$ is to permit both quadratic and cubic reduction of the integral, the eight points denoted by $R(x_1 x_2)$ must be arranged on (C_2) as follows :

Since cubic reduction is possible, three of the points (A , B and C) of R must define a triangle inscribed in (C_2) and circumscribed about (I_2), three more (A' , B' , C') must define another such triangle, while the other two points, (D and E) of R must be points of intersection of (C_2) and (I_2).

Again, since quadratic reduction is also possible, it must be possible to pair

* Bolza, l. c., p. 90.

these eight points so that the joins of the four pairs shall pass through a common point O .

We obtain in all the following essentially different ways of pairing the eight points:

$$\left. \begin{array}{l} \text{I} \quad AA'; BB'; CC'; DE. \\ \text{II} \quad AB; A'B'; CC'; DE. \\ \text{III} \quad AB; A'B'; CD; C'E. \\ \text{IV} \quad AA'; B'C'; BD; CE. \\ \text{V} \quad AA'; BB'; CD; C'E. \end{array} \right\} \quad (46)$$

§4.—*Case I, AA' , BB' , CC' and DE intersect in O .*

For this case the following theorems are proved:

I. If in a cubic involution two triples are such that they form, by taking one point, from each, three couples in quadratic involution, the branch points of the cubic involution form two couples in the quadratic involution. This may be stated thus: If the points of any two triples of a cubic involution are in perspective, the centre of perspectivity is the vertex of the self-conjugate triangle of the conic (C_2) and the involution conic (I_2).

II. Conversely, if we project a triple of the involution from a vertex of the self-conjugate triangle upon the normal conic (C_2), we obtain another triple of the involution.

Thus, being given $\mathfrak{S}$ and the sign of J , which are necessary for the construction of the conics (C_2) and (I_2), we may take for O any vertex of the self-conjugate triangle, we then choose for A any point on (C_2), the two tangents from A to (I_2) determines B and C and projecting A , B and C from O , we obtain other triple A' , B' and C' . This is the desired arrangement.

Notice A is perfectly arbitrarily chosen on (C_2) and O is one of three points; we have thus an infinitude of solutions. In all other cases we have a finite number.

In order that the most general reducible integral, §7, P I, shall specialize to the present case of simultaneous reduction by a cubic and by a quadratic transformation, it is necessary and sufficient that the quadratic $g(x)$ shall be chosen apolar to the quadratic ξ .

§5.—*Case II, AB, A'B', CC' and DE intersect in O.*

Take the point O arbitrarily in the plane, and from it draw the two tangents to (I_2) , touching in K and K' and cutting (C_2) in A and B and A' and B' respectively.

The line KK' is the polar of O with regard to (I_2) , but from a relation* between the coordinates of K and C , the third point of the triple ABC and between K' and C' , we learn that the line CC' is the polar of the point O with regard to a conic (B_2) ;

$$(B_2) \quad \frac{x^2}{l_1} + \frac{y^2}{l_2} + \frac{z^2}{l_3} = 0. \quad (47)$$

But, since CC' is to pass through O , O must be on (B_2) ; since, also, DE is to pass through O , O must be a point of intersection of (B_2) and DE .

We may take any two branch points for DE . Two of the allowable positions for O are

$$\rho x = l_1 \sqrt{l_1 \bar{l}_1}, \quad \rho y = l_2 \sqrt{l_2 \bar{l}_2}, \quad \rho z = \pm i \sqrt{l_3 \bar{l}_3 (l_1 \bar{l}_2 + l_2 \bar{l}_1)}. \quad (48)$$

There will be twelve allowable positions for O .

§6.—*Properties of the Conic (B_2) .*

This conic (B_2) has some peculiar properties, and is closely connected with the involution—

- I. The conic (B_2) passes through the double points of the involution on C_2 .
- II. The conic (B_2) is apolar† to (I_2) .

This, together with I, determines (B_2) completely; (B_2) is also apolar to (C_2) .

III. A triangle inscribed in (C_2) and circumscribed about (I_2) is self-conjugate with regard to (B_2) .

This is the geometrical meaning of II above.

From III follows at once a theorem which we reached otherwise in the last paragraph.

* O. Bolza, l. c. 60 (a).

† Meyer, "Apolarität und Rationale Curven," p. 84.

Draw from any point O two tangents to (I_2) , cutting (C_2) in AB and $A'B'$ respectively; the line, joining C and C' which complete the triples of which AB and $A'B'$ are point pairs, is the polar of O with respect to the conic (B_2) .

IV. The points in which the common tangents to (B_2) and (C_2) touch (C_2) are the four points $\overline{T} = 0$.

§7.—*Case III, $AB, A'B', CD, C'E$ intersect in a point.*

Take arbitrarily the point S in the plane. Join S to E and D , two of the points of intersection of (C_2) and (I_2) ; produce these lines to cut the conic (C_2) again in C' and C respectively; the lines AB and $A'B'$ are thus defined, being the polars of C and C' with regard to the conic (B_2) .

If AB and $A'B'$ meet in O , then, in order to fulfill the conditions of our problem, O and S must coincide.

If we use the branch points

$$(D \text{ and } E) \quad \left\{ \begin{array}{l} \rho x = l_1 \sqrt{l_1 \overline{l}_1} = a, \\ \rho y = l_2 \sqrt{l_2 \overline{l}_2} = b, \\ \rho z = \pm l_3 \sqrt{l_3 \overline{l}_3} = \pm c. \end{array} \right\} \quad (49)$$

and impose the condition that O and S are coincident, we find the only permissible positions for O are the points of intersection of the conics:

$$\left. \begin{array}{l} xz(l_1 - l_3)(bx - ay) = l_1 b(x^2 + y^2 + z^2), \\ yz(l_3 - l_2)(bx - ay) = l_2 a(x^2 + y^2 + z^2). \end{array} \right\} \quad (50)$$

Two of these points are the points D and E themselves, and the other two lie one on each of the tangents to the conic (I_2) at the points D and E .

The integrals arising from this case are, however, degenerate, as the radical is over an expression of the fourth degree only, since in each case one of the triples ABC or $A'B'C'$ is a branch triple including a branch point already in R .

§8.—*Cases IV and V.*

These two cases lead to more complicated results. The positions permissible for O are the points of intersection of curves of higher degree than the

second, and as no very interesting theorems appear in the work, the discussion of these cases will be omitted.

PART III.

§1.—*Involutions containing Two Cubes.*

Up to this point we have supposed the involution to be perfectly general, but let us now consider the special case, where the cubic involution contains two cubes.

As Clebsch has shown,* every such involution may be expressed in the form

$$\lambda_1 f + \lambda_2 Q = 0, \quad (51)$$

where $Q = (f\Delta)$ and $\Delta = (ff)_2$.

Then using the notation of Clebsch, we find

$$\left. \begin{aligned} \mathfrak{S} &= (fQ) = -\frac{1}{2} \Delta^2; & J &= (fQ)_3 = R, \\ H_3 &= \frac{1}{12} R \Delta^2; & \Gamma &\equiv 0; & 3H + J\theta &= -\frac{1}{2} R \Delta^2. \end{aligned} \right\} \quad (52)$$

Hence, by applying method of Part I, §1, we find:

The most general reducible hyperelliptic integral corresponding to this special form of the involution has the form

$$\int \frac{\Delta(x) dx}{\sqrt{\Delta(\lambda_1 f + \lambda_2 Q)(\mu_1 f + \mu_2 Q)}}. \quad (53)$$

The conditions necessary and sufficient for such a special form of the involution are

$$J \neq 0, \quad \Gamma \equiv 0.$$

But we can express the involution (51) in the form

$$\lambda'_1 x_1^3 + \lambda'_2 x_2^3 = 0. \quad (54)$$

Then $\mathfrak{S} = x_1^2 x_2^2$ and $3H + J\mathfrak{S} = \frac{1}{2} x_1^2 x_2^2$, and it follows the hyperelliptic integral

$$\int \frac{x_1 x_2 (x dx)}{\sqrt{x_1 x_2 (\lambda'_1 x_1^3 + \lambda'_2 x_2^3)(\mu'_1 x_1^3 + \mu'_2 x_2^3)}} \quad (55)$$

* L. c., §89.

is reducible to the elliptic integral

$$\int \frac{(y dy)}{\sqrt{y_1 y_2 (\lambda'_1 y_1 + \lambda'_2 y_2)(\mu'_1 y_1 + \mu'_2 y_2)}}$$

by the transformation

$$\begin{aligned} y_1 &= x_1^3, \\ y_2 &= x_2^3. \end{aligned}$$

§2.—*Cases of Simultaneous Reduction of the Second and Third Degree.*

Is it possible that the octavic which occurred in the integral (53)

$$\Delta (\lambda_1 f + \lambda_2 Q)(\mu_1 f + \mu_2 Q)$$

or its normal form

$$x_1 x_2 (\lambda'_1 x_1^3 + \lambda'_2 x_2^3)(\mu'_1 x_1^3 + \mu'_2 x_2^3) \quad (56)$$

can be decomposed in such a way as to yield also a quadratic reduction of the integral?

If this is possible, then the octavic (56) must coincide with some one of the invariant octavics found in the table, Part II, §2.

If $\lambda'_1 : \lambda'_2 = \mu'_2 : -\mu'_1$, then the octavic is invariant under the dihedron group ($n = 3$), and, by making use of our table, this octavic yields us three reducible integrals. If, in particular, $\lambda'_1 : \lambda'_2 = \mu'_2 : -\mu'_1 = 1$, the octavic assumes the form

$$x_1 x_2 (x_1^6 - x_2^6), \quad (57)$$

which is invariant under the dihedron group $n = 6$, and we obtain from the table, Part II, §2, the following three reducible integrals:

$$\int \frac{(x_1^2 - \epsilon^{\frac{2n+1}{3}\pi i} x_2^2)(x dx)}{\sqrt{x_1 x_2 (x_1^6 - x_2^6)}}, \quad \text{for } n = 0, 1, 2. \quad (58)$$

The octavic $\Delta (\lambda_1 f + \lambda_2 Q)(\mu_1 f + \mu_2 Q)$ will be invariant under the dihedron group $n = 6$ if

$$\mu_1 : \mu_2 = -\frac{R}{2} \lambda_2 : \lambda_1,$$

where R has the meaning given in (52).

If, now, we choose our notation so that

$$\lambda_1 f + \lambda_2 Q = f_{\lambda_1 \lambda_2} = f \text{ and } \mu_1 f + \mu_2 Q = Q_{\mu_1 \mu_2} = Q,$$

we have the reducible integrals

$$\int \frac{a_x^2 a_{\alpha_1} (x dx)}{\sqrt{\Delta \cdot Q \cdot f}}, \quad (59)$$

where

$$a_x^2 = f \text{ and } Q = (x\alpha_1)(x\alpha_2)(x\alpha_3)$$

and the theorem :

For a given octavic $\Delta f Q$ we have altogether four reducible integrals, one (53), whose numerator is Δ , by a transformation of the third degree, and three others whose numerators are the first polars of f with regard to the different roots of Q , by transformations of the second degree.

§3.—*Integrals Treated by Method of Periods.*

In the last part of the paper, the reduction problem is studied for a special hyperelliptic curve

$$y^2 = x(1 - x^6) \quad (60)$$

by means of the Weierstrass-Picard theorem on the periods of reducible integrals.

A system of three linearly independent integrals of the first kind, with y for denominators is chosen, a Riemann surface is constructed and a canonical system of cuts made.

The moduli of periodicity of the three integrals at the various cuts are tabulated, and by repeated applications of the Weierstrass-Picard theorem certain reducible integrals are found which all have the denominator $y = \sqrt{x(1-x^6)}$. Among these there are three reducible by quadratic transformation, and are exactly the three given in equation (58), where they were obtained in an entirely different manner. Finally, the following theorem is obtained :

Every integral of the form

$$\begin{aligned} (a_1 + b_1 i) \frac{\Gamma \frac{3}{4}}{\Gamma \frac{1}{4}} \frac{1}{\sqrt{\pi}} \int \frac{x \cdot dx}{\sqrt{x(1-x^6)}} + (a_2 + b_2 i) \frac{\sqrt{2\pi}(1-i)}{\Gamma \frac{5}{12} \Gamma \frac{1}{12}} \int \frac{(1-x^3) dx}{\sqrt{x(1-x^6)}} \\ + (a_3 + b_3 i) \frac{\sqrt{6\pi}(1+i)}{\Gamma \frac{1}{12} \Gamma \frac{5}{12}} e^{\frac{2\pi i}{3}} \int \frac{(x^2 - e^{\frac{5\pi i}{3}}) dx}{\sqrt{x(1-x^6)}} \end{aligned}$$

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(where a_i and b_i are rational), is reducible to an elliptic integral, and there are no other hyperelliptic integrals of the first kind belonging to the equation

$$y^2 = x(1 - x^6)$$

which are reducible by a transformation of any degree.